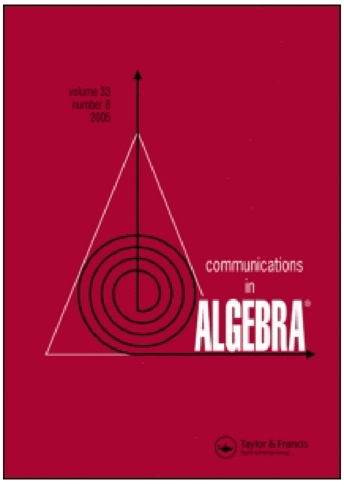

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ON PERFECTLY GENERATING PROJECTIVE CLASSES IN TRIANGULATED CATEGORIES

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We say that a projective class in a triangulated category with coproducts is perfect if the corresponding ideal is closed under coproducts of maps. We study perfect projective classes and the associated phantom and cellular towers. Given a perfect generating projective class, we show that every object is isomorphic to the homotopy colimit of a cellular tower associated to that object. Using this result and the Neeman's Freyd-style representability theorem, we give a new proof of Brown Representability Theorem.

Key Words: Brown representability; Perfect projective class; Triangulated category with coproducts.

2000 Mathematics Subject Classification: 18E15; 18E35; 18E40; 16D90.

INTRODUCTION

The notion of projective classes in pointed categories goes back to Eilenberg and Moore [4]. In this article we consider projective classes in a category $\mathcal{T}$ which is triangulated. In this settings projective classes may be defined as pairs $(\mathcal{P}, \mathfrak{F})$, with $\mathcal{P} \subseteq \mathcal{T}$ a class of objects and $\mathfrak{F} \subseteq \mathcal{T}^{\rightarrow}$ a class of maps (here $\mathcal{T}^{\rightarrow}$ is the category of all maps in $\mathcal{T}$) such that $\mathcal{P}$ is closed under direct factors, $\mathfrak{F}$ is an ideal (that means $\phi, \phi \in \mathfrak{F}$, and $\alpha, \beta \in \mathcal{T}^{\rightarrow}$, implies $\phi + \phi'$, $\alpha\phi\beta \in \mathfrak{F}$, whenever the operations are defined), the composite $p \to x \stackrel{\phi}{\to} x'$ is zero for all $p \in \mathcal{P}$ and all $\phi \in \mathfrak{F}$, and each object $x \in \mathcal{T}$ lies in an exact triangle $\Sigma^{-1}x' \to p \to x \stackrel{\phi}{\to} x'$, with $p \in \mathcal{P}$ and $\phi \in \mathfrak{F}$. Note also that all projective classes which we deal with are stable under suspensions and desuspensions in $\mathcal{T}$. Fix an object $x \in \mathcal{T}$. Choosing repeatedly triangles as above, we construct two towers in $\mathcal{T}$ associated to x, namely the phantom and the cellular tower. The whole construction is similar to the choice of a projective resolution for an object in an abelian category.

Let κ be a regular cardinal. We say that a projective class $(\mathcal{P}, \mathfrak{F})$ is κ-perfect, provided that the ideal $\mathfrak{F}$ is closed under κ-coproducts in $\mathcal{T}^{\rightarrow}$, that is coproducts of less that κ maps, respectively perfect if it is κ-perfect for all cardinals κ. For projective classes which are induced by sets our definition of perfectness is

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equivalent to that of [10], explaining our terminology. Further we say that $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$ if for any $x \in \mathcal{T}$, $\mathcal{T}(\mathcal{P}, x) = 0$ implies $x = 0$. It seems that an important role is played by $\aleph_1$-perfect projective classes, that means projective classes $(\mathcal{P}, \mathfrak{F})$ with $\mathfrak{F}$ closed under countable coproducts. In this case we prove that the homotopy colimit of a tower whose maps belong to $\mathfrak{F}$ is zero (see Lemma 2.2). In particular the homotopy colimit of the phantom tower associated to an object vanishes. If, in addition, we assume that $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$ then Theorem 2.5 tells us that every object x is (isomorphic to) the homotopy colimit of every associated cellular tower. Note also that the hypothesis of $\aleph_1$-perfectness seems to be implicitly assumed by Christensen in [3], as we may see from Proposition 2.3 and Remark 2.4.

Using the product of two projective classes defined in [3], we recall the construction of the nth power $(\mathcal{P}^{*n}, \mathfrak{F}^{*n})$ of a projective class $(\mathcal{P}, \mathfrak{F})$, for $n \in \mathbb{N}$. In [14] it is shown that, if $(\mathcal{P}, \mathfrak{F})$ is induced by a set, then for every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products the comma category $\mathcal{P}^{*n}/F$ has a weak terminal object, for all $n \in \mathbb{N}$. Provided that $(\mathcal{P}, \mathfrak{F})$ is $\aleph_1$-perfect, we use the fact that every x is the homotopy colimit of its cellular tower in order to extend the above property to the whole category $\mathcal{T}/F$. We deduce a version of Brown Representability Theorem for triangulated categories with coproducts which are $\aleph_1$-perfectly generated by a projective class satisfying the additional property that every category $\mathcal{P}^{*n}/F$ has a weak terminal object, for every $n \in \mathbb{N}$ and every cohomological functor which sends coproducts into products $F : \mathcal{T} \to \mathcal{A}b$ (see Theorem 3.7). In particular if the projective class is induced by a set, then this additional property is automatically fulfilled, and we obtain in Corollary 3.8 the version of Brown Representability due to Krause in [8, Theorem A], but our proof is completely different, as it is based on the Freyd–style representability theorem of [14].

For our version of Brown Representability the finite powers of a projective class is all what we need. We still treated the case of transfinite ordinals, following a suggestion of Neeman (see [14, Remark 0.10]). A minor modification of the arguments in [14] shows that if $\mathcal{T} = \mathcal{P}^{*i}$ for some ordinal i, then the Brown representability theorem holds for $\mathcal{T}$. We fill in the details this observation in Lemmas 3.3 and 3.4. On the other hand, if every $x \in \mathcal{T}$ is the homotopy colimit of its $\mathfrak{F}$-cellular tower, then $\mathcal{T} = \mathcal{P}^{*\omega} * \mathcal{P}^{*\omega}$, where ω is the first infinite ordinal. But due to a technical detail we are not able to deduce, as in the case of finite ordinals (see [3, Note 3.6]), that $\mathcal{P}^{*\omega} * \mathcal{P}^{*\omega} = \mathcal{P}^{*(\omega+\omega)}$.

In all categories we consider, the homomorphisms between two objects form a set and not a genuine class. For undefined terms and properties concerning triangulated categories, we refer to [16]. The standard reference for abelian category is [5]. For general theory of categories we refer the reader to [13] or [17].

1. PROJECTIVE CLASSES AND ASSOCIATED TOWERS

Consider a preadditive category $\mathcal{T}$. Then by a *$\mathcal{T}$-module* we understand a functor $X : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$. Such a functor is called *finitely presented* if there is an exact sequence of functors

$$\mathcal{T}(-, y) \to \mathcal{T}(-, x) \to X \to 0$$

for some $x, y \in \mathcal{T}$. Using the Yoneda Lemma, we know that the class of all natural transformations between two $\mathcal{T}$-modules X and Y denoted $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is actually a set, provided that X is finitely presented. We consider the category $\mathrm{mod}(\mathcal{T})$ of all finitely presented $\mathcal{T}$-modules, having as morphisms sets $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ for all $X, Y \in \mathrm{mod}(\mathcal{T})$. The Yoneda functor

$$H = H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T}) \text{ given by } H_{\mathcal{T}}(x) = \mathcal{T}(-, x)$$

is an embedding of $\mathcal{T}$ into $\mathrm{mod}(\mathcal{T})$, according to the Yoneda Lemma. If, in addition, $\mathcal{T}$ has coproducts, then $\mathrm{mod}(\mathcal{T})$ is cocomplete and the Yoneda embedding preserves coproducts. It is also well known (and easy to prove) that, if $F : \mathcal{T} \to \mathcal{A}$ is a functor into an additive category with cokernels, then there is a unique, up to a natural isomorphism, right exact functor $\widehat{F} : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, such that $F = \widehat{F} \circ H_{\mathcal{T}}$ (see [9, Universal property 2.1]). Moreover, F preserves coproducts if and only if $\widehat{F}$ preserves colimits.

In this article, the category $\mathcal{T}$ will be triangulated. Recall that $\mathcal{T}$ is supposed to be additive. A functor $\mathcal{T} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ is called *homological* if it sends triangles into exact sequences. A contravariant functor $\mathcal{T} \to \mathcal{A}$ which is homological regarded as a functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{A}$ is called *cohomological* (see [16, Definition 1.1.7 and Remark 1.1.9]). An example of a homological functor is the Yoneda embedding $H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T})$. We know: $\mathrm{mod}(\mathcal{T})$ is an abelian category, and for every functor $F : \mathcal{T} \to \mathcal{A}$ into an abelian category, the unique right exact functor $\widehat{F} : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$ extending F is exact if and only if F is homological, by [12, Lemma 2.1]. Moreover, $\mathrm{mod}(\mathcal{T})$ is a Frobenius abelian category, with enough injectives and enough projectives, by [16, Corollary 5.1.23]. Injective and projective objects in $\mathrm{mod}(\mathcal{T})$ are, up to isomorphism, exactly objects of the form $\mathcal{T}(-, x)$ for some $x \in \mathcal{T}$, provided that the idempotents in $\mathcal{T}$ split.

From now on, we suppose $\mathcal{T}$ has arbitrary coproducts, so the idempotents in $\mathcal{T}$ split according to [16, Proposition 1.6.8]. First we record some easy but useful results. Recall that a *homotopy colimit* of a tower of objects and maps

$$x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots$$

is defined via the triangle

$$\coprod_{n\in\mathbb{N}} x_n \xrightarrow{1-\phi} \coprod_{n\in\mathbb{N}} x_n \to \operatorname{hocolim} x_n \to \Sigma \coprod_{n\in\mathbb{N}} x_n,$$

where ϕ is the unique morphism which makes commutative all the diagrams of the form

$$\begin{tikzcd} x_n \arrow[r] \arrow[d, "\phi_n"'] & \coprod_{n\in\mathbb{N}} x_n \arrow[d, "\phi"] \\ x_{n+1} \arrow[r] & \coprod_{n\in\mathbb{N}} x_n \end{tikzcd} \quad (n \in \mathbb{N}).$$

Obviously, the homotopy colimit of a tower is unique, up to a non unique isomorphism. We denote sometimes the map ϕ by *shift*, especially if we don't need an explicit notation for the maps in the tower.

The following Lemma is the dual of [2, Lemma 5.8(2)]. Note that we give a more general version, replacing the category $\mathcal{A}b$ (more precisely $\mathcal{A}b^{\mathrm{op}}$) with an abelian AB4 category $\mathcal{A}$, where the derived functors $\operatorname{colim}^{(i)}$ of the colimits are computed in the usual manner, by using homology of a complex. Moreover, [2, Lemma 5.8(1)] is a direct consequence of this dual, together with the exactness of colimits in $\mathcal{A}b$ (that is $\operatorname{colim}^{(1)} = 0$).

Lemma 1.1. *Consider a tower $x_0 \stackrel{\phi_0}{\to} x_1 \stackrel{\phi_1}{\to} x_2 \stackrel{\phi_2}{\to} x_3 \to \cdots$ in $\mathcal{T}$. If $F : \mathcal{T} \to \mathcal{A}$ a homological functor which preserves countable coproducts into an abelian AB4 category $\mathcal{A}$, then we have a Milnor exact sequence*

$$0 \to \operatorname{colim} F(x_n) \to F(\operatorname{hocolim} x_n) \to \operatorname{colim}^{(1)} F(\Sigma x_n) \to 0$$

and $\operatorname{colim}^{(i)} F(x_n) = 0$ *for* $i \geq 2$.

Corollary 1.2. *Consider a tower $x_0 \stackrel{\phi_0}{\to} x_1 \stackrel{\phi_1}{\to} x_2 \stackrel{\phi_2}{\to} x_3 \to \cdots$ in $\mathcal{T}$. If $F : \mathcal{T} \to \mathcal{A}$ is a homological functor, which preserves countable coproducts into an abelian AB4 category, such that $F(\Sigma^i \phi_n) = 0$ for all $i \in \mathbb{Z}$ and all $n \geq 0$, then $F(\operatorname{hocolim} x_n) = 0$.*

Proof. With our hypothesis we have $\operatorname{colim} F(x_n) = 0 = \operatorname{colim}^{(1)} F(\Sigma x_n)$, so $F(\operatorname{hocolim} x_n) = 0$ by the Milnor exact sequence of Lemma 1.1. □

Recall that a pair $(\mathcal{P}, \mathfrak{F})$ consisting of a class of objects $\mathcal{P} \subseteq \mathcal{T}$ and a class of morphisms $\mathfrak{F}$ is called *projective class* if $\Sigma^n(\mathcal{P}) \subseteq \mathcal{P}$ for all $n \in \mathbb{N}$,

$$\mathcal{P} = \{p \in \mathcal{T} \mid \mathcal{T}(p, \phi) = 0 \text{ for all } \phi \in \mathfrak{F}\},$$
$$\mathfrak{F} = \{\phi \in \mathcal{T} \mid \mathcal{T}(p, \phi) = 0 \text{ for all } p \in \mathcal{P}\}$$

and each $x \in \mathcal{T}$ lies in a triangle $\Sigma^{-1}x' \to p \to x \to x'$, with $p \in \mathcal{P}$ and $x \to x'$ in $\mathfrak{F}$ (see [3]). Note that we work only with projective classes which are stable under (de)suspensions; generally, it is possible to define a projective class without this condition. Clearly, $\mathcal{P}$ is closed under coproducts and direct factors, $\mathfrak{F}$ is an ideal, and $\mathfrak{F}$ is stable under (de)suspensions. Moreover, $\mathcal{P}$ and $\mathfrak{F}$ determine each other. A triangle of the form $x \to y \to z \to \Sigma x$ is called $\mathfrak{F}$-*exact* if the morphism $z \to \Sigma x$ belongs to $\mathfrak{F}$. If this is the case, the morphisms $x \to y$ and $y \to z$ are called $\mathfrak{F}$-*monic*, respectively $\mathfrak{F}$-*epic*.

Let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$. The inclusion functor $\varphi : \mathcal{P} \to \mathcal{T}$ induces a unique right exact functor φ^* making commutative the diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi"] \arrow[d, "H_{\mathcal{P}}"'] & \mathcal{T} \arrow[d, "H_{\mathcal{T}}"] \\ \operatorname{mod}(\mathcal{P}) \arrow[r, "\varphi^*"] & \operatorname{mod}(\mathcal{T}), \end{tikzcd}$$

where $H_{\mathcal{P}}$ and $H_{\mathcal{T}}$ are the respective Yoneda functors. More explicitly,

$$\varphi^*(\mathcal{P}(-, p)) = \mathcal{T}(-, p)$$

for all $p \in \mathcal{P}$, and φ^* is right exact. Moreover, since φ is fully-faithful, φ^* has the same property [9, Lemma 2.6].

A *weak kernel* for a morphism $y \to z$ in a preadditive category $\mathcal{C}$ is a morphism $x \to y$ such that, the induced sequence of abelian groups $\mathcal{C}(t, x) \to \mathcal{C}(t, y) \to \mathcal{C}(t, z)$ is exact for all $t \in \mathcal{C}$. Return to the case of a projective class $(\mathcal{P}, \mathfrak{F})$ in the triangulated category $\mathcal{T}$. To construct a weak kernel of a morphism $q \to r$ in $\mathcal{P}$ we proceed as follows: The morphism fits into a triangle $x \to q \to r \to \Sigma x$; let $\Sigma^{-1}x' \to p \to x \to x'$ an $\mathfrak{F}$-exact triangle with $p \in \mathcal{P}$; then the composite map $p \to x \to q$ gives the desired weak kernel. Therefore, $\operatorname{mod}(\mathcal{P})$ is abelian (for example by [9, Lemma 2.2], but this is also well-known). Moreover, the restriction functor

$$\varphi_* : \operatorname{mod}(\mathcal{T}) \to \operatorname{mod}(\mathcal{P}), \qquad \varphi_*(X) = X \circ \varphi \quad \text{for all } X \in \operatorname{mod}(\mathcal{T})$$

is well defined, and it is the exact right adjoint of φ^*, by [8, Lemma 2].

We know by [3, Lemma 3.2] that a pair $(\mathcal{P}, \mathfrak{F})$ is a projective class, provided that $\mathcal{P}$ is a class of objects closed under direct factors, $\mathfrak{F}$ is an ideal, $\mathcal{P}$ and $\mathfrak{F}$ are *orthogonal* (that means, the composite $p \to x \to x'$ is zero for all $p \in \mathcal{P}$ and all $x \to x'$ in $\mathfrak{F}$) and each object $x \in \mathcal{T}$ lies in an $\mathfrak{F}$-exact triangle $\Sigma^{-1}x' \to p \to x \to x'$, with $p \in \mathcal{P}$. If $\mathcal{S}$ is a set of objects in $\mathcal{T}$, then $\operatorname{Add}\mathcal{S}$ denotes, as usual, the class of all direct factors of arbitrary coproducts of objects in $\mathcal{S}$. The following lemma is straightforward (see also [3, Definition 5.2 and the following paragraph]).

Lemma 1.3. *Consider a set $\mathcal{S}$ of objects in $\mathcal{T}$ which is closed under suspensions and desuspensions. Denote by $\mathcal{P} = \operatorname{Add}\mathcal{S}$, and let $\mathfrak{F}$ be the class of all morphisms ϕ in $\mathcal{T}$ such that $\mathcal{T}(s, \phi) = 0$ for all $s \in \mathcal{S}$. Then $(\mathcal{P}, \mathfrak{F})$ is a projective class.*

We will say that the projective class $(\mathcal{P}, \mathfrak{F})$ given in Lemma 1.3 is *induced* by the set $\mathcal{S}$. Note also that if $\mathcal{S}$ is an essentially small subcategory of $\mathcal{T}$, such that $\Sigma^n(\mathcal{S}) \subseteq \mathcal{S}$ for all $n \in \mathbb{Z}$, then we will also speak about the projective class induced by $\mathcal{S}$, understanding the projective class induced by a representative set of isomorphism classes of objects in $\mathcal{S}$. If, in particular, κ is a regular cardinal, $\mathcal{S}$ consists of κ-small objects and it is closed under coproducts of less than κ objects (for example, if $\mathcal{S}$ is the subcategory of all κ-compact object of $\mathcal{T}$), then $\operatorname{mod}(\mathcal{P})$ is equivalent to the category of all functors $\mathcal{S}^{\mathrm{op}} \to \mathcal{A}b$ which preserve products of less than κ objects, by [11, Lemma 2], category used extensively in [16] as a locally presentable approximation of $\operatorname{mod}(\mathcal{T})$.

Remark 1.4. Under the hypotheses of Lemma 1.3, a map $x \to y$ in $\mathcal{T}$ is $\mathfrak{F}$-monic ($\mathfrak{F}$-epic) if and only if the induced map $\mathcal{T}(s, x) \to \mathcal{T}(s, y)$ injective (respectively, surjective) for all $s \in \mathcal{S}$.

As in [2, 3], given a projective class $(\mathcal{P}, \mathfrak{F})$ in $\mathcal{T}$, we construct two towers of morphisms associated to each $x \in \mathcal{T}$ as follows: We denote $x_0 = \Sigma^{-1}x$. Inductively, if $x_n \in \mathcal{T}$ is given, for $n \in \mathbb{N}$, then there is an $\mathfrak{F}$-exact triangle

$$\Sigma^{-1}x_{n+1} \to p_n \to x_n \xrightarrow{\phi_n} x_{n+1}$$

in $\mathcal{T}$, by definition of a projective class. Consider then the tower:

$$\Sigma^{-1}x = x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots .$$

Such a tower is called a $\mathfrak{F}$-*phantom tower* of x. The explanation of the terminology is that morphisms ϕ_n belong to $\mathfrak{F}$ for all $n \in \mathbb{N}$, and $\mathfrak{F}$ may be thought as a generalization of the ideal of classical phantom maps in a compactly generated triangulated category. (Clearly, $\mathfrak{F}$ coincides with the ideal of classical phantom maps, provided that the projective class $(\mathcal{P}, \mathfrak{F})$ is induced by the full essentially small subcategory consisting of all compact objects.)

Observe that there are more $\mathfrak{F}$-phantom towers associated to the same element $x \in \mathcal{T}$, according with the choices of the $\mathfrak{F}$-epic map $p_n \to x_n$ at each step $n \in \mathbb{N}$. The analogy with projective resolutions in abelian categories is obvious.

Choose an $\mathfrak{F}$-phantom tower of $x \in \mathcal{T}$ as in the definition above. We denote by ϕ^n the composed map $\phi_{n-1} \ldots \phi_1\phi_0 : \Sigma^{-1}x \to x_n$, for all $n \in \mathbb{N}^*$, and we set $\phi^0 = 1_{\Sigma^{-1}x}$. Then let x^n be defined, uniquely up to a nonunique isomorphism, by the triangle $\Sigma^{-1}x \xrightarrow{\phi^n} x_n \to x^n \to x$. The octahedral axiom allows us to complete the commutative diagram

$$\begin{tikzcd}
 & p_n \arrow[r, equal] \arrow[d] & p_n \arrow[d] & \\
\Sigma^{-1}x \arrow[r, "\phi^n"] \arrow[d, equal] & x_n \arrow[r] \arrow[d, "\phi_n"] & x^n \arrow[r] \arrow[d] & x \arrow[d, equal] \\
\Sigma^{-1}x \arrow[r, "\phi^{n+1}"] & x_{n+1} \arrow[r] \arrow[d] & x^{n+1} \arrow[r] \arrow[d] & x \\
 & \Sigma p_n \arrow[r, equal] & \Sigma p_n &
\end{tikzcd}$$

with the triangle in the second column.

Therefore, we obtain an another tower of objects

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots ,$$

where for each $n \in \mathbb{N}$ we have a triangle $p_n \to x^n \to x^{n+1} \to \Sigma p_n$, with $p_n \in \mathcal{P}$ chosen in the construction of the above $\mathfrak{F}$-phantom tower. Such a tower is called a $\mathfrak{F}$-*cellular tower* of $x \in \mathcal{T}$.

Considering homotopy colimits of the $\mathfrak{F}$-phantom and $\mathfrak{F}$-cellular towers, we obtain a sequence

$$\Sigma^{-1}x \to \operatorname{hocolim} x_n \to \operatorname{hocolim} x^n \to x.$$

It is not known whether the induced sequence can be chosen to be a triangle (see [2, p. 302]). However, the answer to this question is yes, provided that $\mathcal{T}$ is the homotopy category of a suitable stable closed model category in the sense of [7].

Proposition 1.5. *Let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$, and let denote by $\varphi : \mathcal{P} \to \mathcal{T}$ the inclusion functor. For every $x \in \mathcal{T}$, we consider an $\mathfrak{F}$-phantom tower and an $\mathfrak{F}$-cellular tower as above. Then we have an exact sequence*

$$0 \to \coprod(\varphi_* \circ H_{\mathcal{T}})(x^n) \xrightarrow{1-shift} \coprod(\varphi_* \circ H_{\mathcal{T}})(x^n) \to (\varphi_* \circ H_{\mathcal{T}})(x) \to 0,$$

where $\varphi_* : \operatorname{mod}(\mathcal{T}) \to \operatorname{mod}(\mathcal{P})$ is the restriction functor. Consequently,

$$\operatorname{colim}(\varphi_* \circ H_{\mathcal{T}})(x^n) \cong (\varphi_* \circ H_{\mathcal{T}})(x) \quad \text{and} \quad \operatorname{colim}^{(1)}(\varphi_* \circ H_{\mathcal{T}})(x^n) = 0.$$

Proof. By applying the functor $\varphi_* \circ H_{\mathcal{T}}$ to the diagram above defining an $\mathfrak{F}$-cellular tower associated to x, we obtain a commutative diagram in the abelian category with coproducts $\operatorname{mod}(\mathcal{P})$:

$$\begin{tikzcd}
0 \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x_n) \arrow[r] \arrow[d, "0"'] & (\varphi_* \circ H_{\mathcal{T}})(x^n) \arrow[r] \arrow[d] & (\varphi_* \circ H_{\mathcal{T}})(x) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x_{n+1}) \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x^{n+1}) \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x) \arrow[r] & 0 .
\end{tikzcd}$$

The conclusion follows by [10, Lemma 7.1.2]. □

2. PERFECTLY GENERATING PROJECTIVE CLASSES

Consider a cardinal κ. Recall that κ is said to be *regular* provided that it is infinite and it cannot be written as a sum of less than κ cardinals, all smaller than κ. By *κ-coproducts* we understand coproducts of less that κ-objects.

Proposition 2.1. *Let κ be a regular cardinal and let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$. Denote by $\varphi : \mathcal{P} \to \mathcal{T}$ the inclusion functor. Then the functor $\varphi_* : \operatorname{mod}(\mathcal{T}) \to \operatorname{mod}(\mathcal{P})$, $\varphi_*(X) = X \circ \varphi$ preserves κ-coproducts if and only if $\mathfrak{F}$ is closed under κ-coproducts (of maps).*

Proof. The exact functor φ_* having a fully-faithful left adjoint induces an equivalence $\operatorname{mod}(\mathcal{T})/\operatorname{Ker}\varphi_* \to \operatorname{mod}(\mathcal{P})$. Since $\operatorname{mod}(\mathcal{T})$ is AB4, we know that φ_* preserves κ-coproducts if and only if $\operatorname{Ker}\varphi_*$ is closed under κ-coproducts. Obviously $\mathfrak{F} = \{\phi \mid (\varphi_* \circ H_{\mathcal{T}})(\phi) = 0\}$. Using the proof of [12, Section 3], we observe that

$$\operatorname{Ker}\varphi_* = \{X \in \operatorname{mod}(\mathcal{T}) \mid X \cong \operatorname{im} H_{\mathcal{T}}(\phi) \text{ for some } \phi \in \mathfrak{F}\}.$$

Now suppose $\mathfrak{F}$ to be closed under κ-coproducts, and let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of objects in $\operatorname{Ker}\varphi_*$, with the cardinality less than κ. Thus $M_\lambda \cong \operatorname{im} H_{\mathcal{T}}(\phi_\lambda)$ for some $\phi_\lambda \in \mathfrak{F}$, for all $\lambda \in \Lambda$. Therefore, using again condition AB4 (coproducts in $\operatorname{mod}(\mathcal{T})$ are exact, so they commute with images), we obtain:

$$\coprod_{\lambda\in\Lambda} M_\lambda \cong \coprod_{\lambda\in\Lambda} \operatorname{im} H_{\mathcal{T}}(\phi_\lambda) \cong \operatorname{im}\left(\coprod_{\lambda\in\Lambda} H_{\mathcal{T}}(\phi_\lambda)\right) \cong \operatorname{im} H_{\mathcal{T}}\left(\coprod_{\lambda\in\Lambda} \phi_\lambda\right),$$

showing that $\coprod_{\lambda\in\Lambda} M_\lambda \in \operatorname{Ker}\varphi_*$.

Conversely, if $\operatorname{Ker}\varphi_*$ is closed under κ-coproducts, and $\{\phi_\lambda \mid \lambda \in \Lambda\}$ is a set of maps in $\mathfrak{F}$, with the cardinality less than κ, then

$$\varphi_*\left(\operatorname{im} H_{\mathcal{T}}\left(\coprod_{\lambda\in\Lambda} \phi_\lambda\right)\right) = \varphi_*\left(\coprod_{\lambda\in\Lambda} \operatorname{im} H_{\mathcal{T}}(\phi_\lambda)\right) = 0,$$

so $\mathfrak{F}$ is closed under κ-coproducts. □

We call *κ-perfect* the projective class $(\mathcal{P}, \mathfrak{F})$ if the equivalent conditions of Proposition 2.1 hold true. The projective class will be called *perfect* if it is κ-perfect for all regular cardinals κ, that is, $\mathfrak{F}$ is closed under arbitrary coproducts. Following [3], we say that a projective class $(\mathcal{P}, \mathfrak{F})$ *generates* $\mathcal{T}$ if for any $x \in \mathcal{T}$, we have $x = 0$ provided that $\mathcal{T}(p, x) = 0$, for each $p \in \mathcal{P}$. Immediately, we can see that $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$ if and only if $\varphi \circ H_{\mathcal{T}} : \mathcal{T} \to \operatorname{mod}(\mathcal{P})$ reflects isomorphisms, that is, if $\alpha : x \to y$ is a morphism in $\mathcal{T}$ such that the induced morphism $(\varphi \circ H_{\mathcal{T}})(\alpha)$ is an isomorphism in $\operatorname{mod}(\mathcal{P})$, then α is an isomorphism in $\mathcal{T}$, where $\varphi : \mathcal{P} \to \mathcal{T}$ denotes, as usual, the inclusion functor. Another equivalent statement is $\mathfrak{F}$ does not contain nonzero identity maps. Consider now an essentially small subcategory $\mathcal{S}$ of $\mathcal{T}$ which is closed under suspensions and desuspensions, and $(\mathcal{P}, \mathfrak{F})$ the projective class induced by $\mathcal{S}$. Since coproducts of triangles are triangles, we conclude by Remark 1.4 that $\mathfrak{F}$ is closed under coproducts exactly if $\mathcal{S}$ satisfies the following condition: If $x_i \to y_i$ with $i \in I$ is a family of maps, such that $\mathcal{T}(s, x_i) \to \mathcal{T}(s, y_i)$ is surjective for all $i \in I$, then the induced map $\mathcal{T}(s, \coprod x_i) \to \mathcal{T}(s, \coprod y_i)$ is also surjective. Thus $(\mathcal{P}, \mathfrak{F})$ perfectly generates $\mathcal{T}$ in the sense above if and only if $\mathcal{S}$ perfectly generates $\mathcal{T}$ in the sense given in [10, Section 5] (see also [8] for a version relativized at the cardinal $\kappa = \aleph_1$).

Lemma 2.2. *Consider a tower* $x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots$ *in* $\mathcal{T}$. *If* $(\mathcal{P}, \mathfrak{F})$ *is an* $\aleph_1$*-perfect projective class in* $\mathcal{T}$ *and* $\phi_n \in \mathfrak{F}$ *for all* $n \geq 0$, *then* $\operatorname{hocolim} x_n = 0$.

Proof. We apply Corollary 1.2 to the homological functor, which preserves countable coproducts $\varphi_* \circ H_T : \mathcal{T} \to \operatorname{mod}(\mathcal{P})$, where $\varphi : \mathcal{P} \to \mathcal{T}$ is the inclusion functor. □

Proposition 2.3. *If* $(\mathcal{P}, \mathfrak{F})$ *is a* $\aleph_1$*-perfect projective class in* $\mathcal{T}$, *then a necessary and sufficient condition for* $(\mathcal{P}, \mathfrak{F})$ *to generate* $\mathcal{T}$ *is*

$$\lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) = 0 = \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y),$$

for all $x, y \in \mathcal{T}$ and any choice

$$x = x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots,$$

of an $\mathfrak{F}$-phantom tower of x. Here by $\lim^{(1)}$ *we understand the first derived functor of the limit.*

Proof. Let show the sufficiency first. If $x \in \mathcal{T}$ has the property $\mathcal{T}(p, x) = 0$ for all $p \in \mathcal{P}$, then $1_x \in \mathfrak{F}$ and a $\mathfrak{F}$-phantom tower of x is

$$x = x_0 \xrightarrow{1_x} x_1 = x \xrightarrow{1_x} x_2 = x \to \cdots.$$

Then $0 = \lim_{n\in\mathbb{N}} \mathcal{T}(x_n, x) = \mathcal{T}(x, x)$, so $x = 0$.

Now we show the necessity. Let $x, y \in \mathcal{T}$, and consider an $\mathfrak{F}$-phantom tower of x as above. Applying the functor $\mathcal{T}(-, y)$ to this tower, we obtain a sequence of abelian groups

$$\mathcal{T}(x, y) = \mathcal{T}(x_0, y) \xleftarrow{(\phi_0)_*} \mathcal{T}(x_1, y) \xleftarrow{(\phi_1)_*} \mathcal{T}(x_2, y) \xleftarrow{(\phi_2)_*} \mathcal{T}(x_3, y) \leftarrow \cdots.$$

Computing the derived functors of the limit of such a sequence in the usual manner, we know that $\lim^{(n)}$ is zero for $n \geq 2$ and lim, $\lim^{(1)}$ are given by the exact sequence

$$0 \to \lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \to \prod_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \xrightarrow{(1-\phi)_*} \prod_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \to \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y) \to 0,$$

where $\phi : \coprod_{n\in\mathbb{N}} x_n \to \coprod_{n\in\mathbb{N}} x_n$ is constructed as in the definition of the homotopy colimit. Applying $\mathcal{T}(p, -)$ to the commutative squares which define ϕ, we obtain also commutative squares

$$\begin{tikzcd}
\mathcal{T}(p, x_n) \arrow[r] \arrow[d, "0=\mathcal{T}(p,\phi_n)"'] & \mathcal{T}(p, \coprod_{n\in\mathbb{N}} x_n) \arrow[d, "\mathcal{T}(p,\phi)"] \\
\mathcal{T}(p, x_{n+1}) \arrow[r] & \mathcal{T}(p, \coprod_{n\in\mathbb{N}} x_n)
\end{tikzcd} \qquad (n \in \mathbb{N}),$$

for all $p \in \mathcal{P}$. According to Proposition 2.1, the $\aleph_1$-perfectness of $(\mathcal{P}, \mathfrak{F})$ means that $\mathcal{T}(-, \coprod_{n\in\mathbb{N}} x_n)|_{\mathcal{P}}$ is the coproduct in $\mathrm{mod}(\mathcal{P})$ of the set

$$\{\mathcal{T}(-, x_n)|_{\mathcal{P}} | n \in \mathbb{N}\},$$

thus we deduce $\mathcal{T}(p, \phi) = 0$. Now $\mathcal{T}(p, 1 - \phi) = \mathcal{T}(p, 1) - \mathcal{T}(p, \phi) = \mathcal{T}(p, 1)$ is an isomorphism, for all $p \in \mathcal{P}$, so $1 - \phi$ is an isomorphism, because $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$. Consequently,

$$\lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) = 0 = \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y).$$

□

Remark 2.4. The hypotheses of Proposition 2.3 are almost identical with those of [3, Proposition 4.4], except the fact that we require, in addition, the $\aleph_1$-perfectness for $(\mathcal{P}, \mathfrak{F})$. Moreover, the conclusion of [3, Proposition 4.4] (namely, the Adams spectral sequence abutting $\mathcal{T}(x, y)$ is conditionally convergent) is equivalent to our conclusion (lim and $\lim^{(1)}$ to be zero). The proofs are also almost identical. Despite that, we have given a detailed proof, because, without our additional condition, we do not see how we can conclude, with our notations, that $\mathcal{T}(p, \phi) = 0$. Thus we fill a gap existing in the proof of [3, Proposition 4.4], due to the missing assumption of $\aleph_1$-perfectness. On the other hand, we do not have a counterexample showing that the conclusion cannot be inferred without this assumption, so the problem is open. Note also that the terms of the Adams spectral sequence of [3] do not depend, for sufficiently large indices, of the choice of the $\mathfrak{F}$-projective resolution of $x \in \mathcal{T}$, so the conclusion of Proposition 2.3 may be formulated simply: The Adams spectral sequence abutting $\mathcal{T}(x, y)$ is conditionally convergent, for any two $x, y \in \mathcal{T}$.

Theorem 2.5. *Let $(\mathcal{P}, \mathfrak{F})$ be an $\aleph_1$-perfectly generating projective class in $\mathcal{T}$. Then for every $x \in \mathcal{T}$, and every choice*

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

of an $\mathfrak{F}$-cellular tower for x, we have $\operatorname{hocolim} x^n \cong x$.

Proof. The homotopy colimit of the $\mathfrak{F}$-cellular tower above is constructed via triangle

$$\coprod_{n\in\mathbb{N}} x^n \xrightarrow{1-shift} \coprod_{n\in\mathbb{N}} x^n \to \operatorname{hocolim} x^n \to \Sigma \coprod_{n\in\mathbb{N}} x^n.$$

We apply to this triangle the homological functor $\varphi_* \circ H_{\mathcal{T}}$ which commutes with countable coproducts. Comparing the resulting exact sequence with the exact sequence given by Proposition 1.5, we obtain a unique isomorphism

$$(\varphi_* \circ H_{\mathcal{T}})(\operatorname{hocolim} x^n) \to (\varphi_* \circ H_{\mathcal{T}})(x),$$

which must be induced by the map $\operatorname{hocolim} x^n \to x$. The generating hypothesis tells us that $\operatorname{hocolim} x^n \cong x$. □

Recall that $\aleph_1$-localizing subcategory of $\mathcal{T}$ means triangulated and closed under countable coproducts.

Corollary 2.6. *If $(\mathcal{P}, \mathfrak{F})$ is an $\aleph_1$-perfectly generating projective class in $\mathcal{T}$, then $\mathcal{T}$ is the smallest $\aleph_1$-localizing subcategory of $\mathcal{T}$, which contains $\mathcal{P}$.*

Proof. Let $x \in \mathcal{T}$, and let

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

be an $\mathfrak{F}$-cellular tower for x. Since for every $n \geq 0$, there exits a triangle $p_n \to x_n \to x_{n+1} \to \Sigma p_n$, with $p_n \in \mathcal{P}$ (see the definition of an $\mathfrak{F}$-cellular tower), we may see inductively that x_n belongs to the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{P}$. Now $\operatorname{hocolim} x^n$ belongs to the smallest $\aleph_1$-localizing subcategory of $\mathcal{T}$ which contains $\mathcal{P}$, and the conclusion follows by Theorem 2.5. □

Remark 2.7. Let $(\mathcal{P}, \mathfrak{F})$ be an $\aleph_1$-perfectly generating projective class in $\mathcal{T}$, and $x \in \mathcal{T}$. If we chose an $\mathfrak{F}$-phantom tower

$$\Sigma^{-1}x = x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots$$

and an $\mathfrak{F}$-cellular tower

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

for x, then $\operatorname{hocolim} x_n = 0$ by Lemma 2.2, and $\operatorname{hocolim} x^n \cong x$ by Theorem 2.5. Thus the triangle $\Sigma^{-1}x \to \operatorname{hocolim} x_n \to \operatorname{hocolim} x^n \to x$ is trivially exact.

Remark 2.8. A filtration analogous to that of Theorem 2.5, for the case of well-generated triangulated categories may be found in [16, Lemma B 1.3].

3. BROWN REPRESENTABILITY VIA PERFECT PROJECTIVE CLASSES

For two projective classes $(\mathcal{P}, \mathfrak{F})$ and $(\mathcal{Q}, \mathfrak{G})$, we define the *product* by

$$\mathcal{P} * \mathcal{Q} = \operatorname{add}\{x \in \mathcal{T} \mid \text{there is a triangle } q \to x \to p \to \Sigma q \text{ with } p \in \mathcal{P}, q \in \mathcal{Q}\},$$

and $\mathfrak{F} * \mathfrak{G} = \{\phi\psi \mid \phi \in \mathfrak{F}, \psi \in \mathfrak{G}\}$. Generally, by add we understand the closure under finite coproducts and direct factors. Since in our case the closure under arbitrary coproducts is automatically fulfilled, add means here simply the closure under direct factors. Thus $(\mathcal{P} * \mathcal{Q}, \mathfrak{F} * \mathfrak{G})$ is a projective class, by [3, Proposition 3.3].

If $(\mathcal{P}_i, \mathfrak{F}_i)$ for $i \in I$ is a family of projective classes, then

$$\left(\operatorname{Add}\left(\bigcup_I \mathcal{P}_i\right), \bigcap_I \mathfrak{F}_i\right)$$

is also a projective class by [3, Proposition 3.1], called the *meet* of the above family.

In a straightforward manner we may use the octahedral axiom in order to show that the product defined above is associative. We may also observe without difficulties that the product of two (respectively the meet of a family of) κ-perfect projective classes is also κ-perfect, where κ is an arbitrary regular cardinal.

Consider now a projective class $(\mathcal{P}, \mathfrak{F})$ in $\mathcal{T}$. We define inductively $\mathcal{P}^{*0} = \{0\}$, $\mathfrak{F}^{*0} = \mathcal{T}^{\to}$ and $\mathcal{P}^{*i} = \mathcal{P} * \mathcal{P}^{*(i-1)}$, $\mathfrak{F}^{*i} = \mathfrak{F} * \mathfrak{F}^{*(i-1)}$, for every non-limit ordinal $i > 0$. If i is a limit ordinal then $(\mathcal{P}^{*i}, \mathfrak{F}^{*i})$ is defined as the meet of all $(\mathcal{P}^{*j}, \mathfrak{F}^{*j})$ with $j < i$. Therefore, $(\mathcal{P}^{*i}, \mathfrak{F}^{*i})$ is a projective class for every ordinal i, which is called the *ith power* of the projective class of $(\mathcal{P}, \mathfrak{F})$ (see also [3], for the case of ordinals less or equal to the first infinite ordinal). Clearly, we have $\mathcal{P}^{*j} \subseteq \mathcal{P}^{*i}$, for all ordinals $j \leq i$.

Remark 3.1. We can inductively see that for $x \in \mathcal{T}$ it holds $x^n \in \mathcal{P}^{*n}$ for all $n \in \mathbb{N}$, where x^n is the nth term of an $\mathfrak{F}$-cellular tower of x.

For example, if $\mathcal{T}$ is compactly generated, and $\mathcal{T}^c$ denotes the subcategory of all compact objects, then the projective class induced by $\mathcal{T}^c$ is obviously perfect, thus we obtain immediate consequence of Theorem 2.5:

Corollary 3.2 ([2, Corollary 6.9]). *If $\mathcal{T}$ is compactly generated then any object $x \in \mathcal{T}$ is the homotopy colimit* $\operatorname{hocolim} x^n$ *of a tower $x^0 \to x^1 \to \cdots$, where $x^n \in \operatorname{Add}(\mathcal{T}^c)^{*n}$, for all $n \in \mathbb{N}$.*

Consider a contravariant functor $F : \mathcal{T} \to \mathcal{A}b$. For a full subcategory $\mathcal{C}$ of $\mathcal{T}$, we consider the comma category $\mathcal{C}/F$ with the objects being pairs of the form (x, a), where $x \in \mathcal{C}$ and $a \in F(x)$, and maps

$$(\mathcal{C}/F)((x, a)(y, b)) = \{\alpha \in \mathcal{T}(x, y) \mid F(\alpha)(b) = a\}.$$

Motivated by [14] it is interesting to find weak terminal objects in $\mathcal{T}/F$, that is objects $(t, b) \in \mathcal{T}/F$, such that for every $(x, a) \in \mathcal{T}/F$ there is a map $(x, a) \to (t, b) \in (\mathcal{T}/F)^{\to}$. Another equivalent formulation of this fact is that the natural transformation $\mathcal{T}(-, t) \to F$ which corresponds under the Yoneda isomorphism to $b \in F(t)$ is an epimorphism. The statement a) of the following lemma is proved by the same argument as [14, Lemma 2.3]. We include a sketch of the proof for the convenience of the reader.

Lemma 3.3. *Let $F : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor which sends coproducts into products.*

a) *If $(\mathcal{P}, \mathfrak{F})$ and $(\mathcal{Q}, \mathfrak{G})$ are projective classes in $\mathcal{T}$ such that $(\mathcal{P}, \mathfrak{F})$ is induced by a set and $\mathcal{Q}/F$ has a weak terminal object, then $(\mathcal{P} * \mathcal{Q})/F$ has a weak terminal object.*
b) *If $(\mathcal{P}_i, \mathfrak{F}_i)$, $i \in I$ are projective classes in $\mathcal{T}$ with the meet $(\mathcal{P}, \mathfrak{F})$, and $\mathcal{P}_i/F$ has a weak terminal object for all $i \in I$, then $\mathcal{P}/F$ has a weak terminal object.*

Proof. a) Let (q, d) be a weak terminal object in $\mathcal{Q}/F$, and let $\mathcal{S}$ be a set which induces the projective class $(\mathcal{P}, \mathfrak{F})$. Obviously, $\mathcal{P}/F$ has a weak terminal object (p, c). Consider an object $(y, a) \in (\mathcal{P} * \mathcal{Q})/F$. Thus there is a triangle $x \to y \to z \to \Sigma x$, with $x \in \mathcal{Q}$ and $z \in \mathcal{P}$. We have $z \amalg z' = \coprod_{i\in I} s_i$ for some $z' \in \mathcal{T}$. We construct the commutative diagram in $\mathcal{T}$ whose rows are triangles

$$\begin{tikzcd}
x \arrow[r] \arrow[d, equal] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & \Sigma x \arrow[d, equal] \\
x \arrow[r, "\alpha"] \arrow[d, "f"'] & y \amalg z' \arrow[r] \arrow[d, "g"] & \coprod_{i\in I} s_i \arrow[r] \arrow[d, equal] & \Sigma x \arrow[d] \\
q \arrow[r, "\beta"'] & y_1 \arrow[r] & \coprod_{i\in I} s_i \arrow[r, "\gamma"'] & \Sigma q
\end{tikzcd}$$

We proceeded as follows: The triangle on the second row is obtained as the coproduct of the initial one with $0 \to z' \to z' \to 0$, and the maps are the canonical injections. For $d' = F(\alpha)(a, 0) \in F(x)$, there is a map $f : (x, d') \to (q, d) \in (\mathcal{Q}/F)^{\to}$, since (q, d) is weak terminal. The first bottom square of the diagram above is homotopy push-out (see [16, Definition 1.4.1 and Lemma 1.4.4]). Clearly $y_1 \in \mathcal{P} * \mathcal{Q}$. Since F is cohomological, there is $a_1 \in F(y_1)$ such that $F(\beta)(a_1) = d$ and $F(g)(a_1) = (a, 0)$. So if we find a map $(y_1, a_1) \to (t, b) \in ((\mathcal{P} * \mathcal{Q})/F)^{\to}$ for a fixed object (t, b), then the conclusion follows.

If we denote by $J \subseteq \bigcup_{s\in\mathcal{S}} \mathcal{T}(s, \Sigma q)$ the set of all maps $s_i \to \coprod_{i\in I} s_i \to \Sigma q$, then γ factors as $\coprod_{i\in I} s_i \xrightarrow{\nabla} \coprod_{s\in J} s \to \Sigma q$, where ∇ is a split epimorphism. Hence the fibre of γ is isomorphic to $y_J \amalg z''$, for some $z'' \in \mathcal{P}$ and y_J defined as the fibre of the canonical map $\coprod_{s\in J} s \to \Sigma q$. Therefore, (y, a) maps to $(t, b) = (t' \amalg p, (b', c))$ where

$$(t', b') = \left(\coprod_{J \subseteq \bigcup_{s\in\mathcal{S}} \mathcal{T}(s,\Sigma q)} \left(\coprod_{u\in F(y_J)} (y_J, u) \right)\right),$$

so the object (t, b) is weak terminal in $(\mathcal{P} * \mathcal{Q})/F$.

b) If $(t_i, a_i) \in \mathcal{P}_i/F$ is a weak terminal object, then $(\coprod_{i\in I} t_i, (a_i)_{i\in I})$ is a weak terminal object in $\mathcal{P}/F$. $\square$

By transfinite induction we obtain the following lemma.

Lemma 3.4. *Let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$ which is induced by a set. For every ordinal i and every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products, the category $\mathcal{P}^{*i}/F$ has a weak terminal object.*

Remark 3.5. For finite ordinals, Lemma 3.4 is the same as [14, Lemma 2.3]. Note also that Neeman defined the operation $*$ without to assume the closure under direct factors, but for a subcategory $\mathcal{C}$ of $\mathcal{T}$ such that (t, b) is weak terminal in $\mathcal{C}/F$, the same object is weak terminal in $\operatorname{add}\mathcal{C}/F$ too.

Proposition 3.6. *Let $(\mathcal{P}, \mathfrak{F})$ be an $\aleph_1$-perfectly generating projective class in $\mathcal{T}$, and let $F : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor which sends coproducts into products. Suppose also that every category $\mathcal{P}^{*n}/F$ has a weak terminal object (t^n, b_n), for $n \in \mathbb{N}$. Then $\mathcal{T}/F$ has a weak terminal object.*

Proof. Denote by I the set of all towers $0 = t^0 \xrightarrow{\tau_0} t^1 \xrightarrow{\tau_1} t^2 \to \cdots$, satisfying $F(\tau_n)(b_{n+1}) = b_n$, for all $n \in \mathbb{N}$. The set I is not empty since for all $n \in \mathbb{N}$, we have $t^n \in \mathcal{P}^{*n} \subseteq \mathcal{P}^{*(n+1)}$ and (t^{n+1}, b_{n+1}) is weak terminal in $\mathcal{P}^{*(n+1)}/F$. Denote also by t_i the homotopy colimits of the tower $i \in I$, and chose $b_i \in F(t_i)$ an element which maps into $(b_n)_{n\in\mathbb{N}}$ via the surjective (see the dual of Lemma 1.1) map $F(t_i) \to \lim_{n\in\mathbb{N}} F(t^n)$. We claim that

$$(t, b) = \left(\coprod_{i\in I} t_i, (b_i)_{i\in I} \right) \in \mathcal{T}/F$$

is a weak terminal object.

In order to prove our claim, let $x \in \mathcal{T}$. As we have seen in Theorem 2.5, it is isomorphic to the the homotopy colimit of its $\mathfrak{F}$-cellular tower $0 = x^0 \xrightarrow{\alpha_0} x^1 \xrightarrow{\alpha_1} x^2 \to \cdots$, associated with a choice of an $\mathfrak{F}$-phantom tower. Thus consider the commutative diagram, whose rows are exact by Lemma 1.1 and whose vertical arrows are induced by the natural transformation corresponding to $b \in F(t)$ via the Yoneda isomorphism

$$\begin{tikzcd} 0 \arrow[r] & \lim^{(1)} \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & \mathcal{T}(x, t) \arrow[r] \arrow[d] & \lim \mathcal{T}(x^n, t) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \lim^{(1)} F(\Sigma x^n) \arrow[r] & F(x) \arrow[r] & \lim F(x^n) \arrow[r] & 0 \end{tikzcd}.$$

If we would prove that the two extreme vertical arrows are surjective, then the middle arrow enjoys the same property and our work would be done.

For $n \in \mathbb{N}$, we know that $\Sigma x^n \in \mathcal{P}^{*n}$ and (t^n, b_n) is weak terminal in $\mathcal{P}^{*n}$, so there is a map $(\Sigma x^n, a_n) \to (t^n, b_n) \in (\mathcal{P}^{*n}/F)^{\to}$ for every element $a_n \in F(\Sigma x^n)$. Because $I \neq \emptyset$, there exists $i \in I$, hence we obtain a map

$$(\Sigma x^n, a_n) \to (t^n, b_n) \to (t_i, b_i) \to (t, b) \in (\mathcal{T}/F)^{\to}$$

showing that the natural map $\mathcal{T}(\Sigma x^n, t) \to F(\Sigma x^n)$ is surjective. Therefore, the first vertical map in the commutative diagram above is surjective as we may see from the following commutative diagram with exact rows:

$$\begin{tikzcd} \prod \mathcal{T}(\Sigma x^n, t) \arrow[r, "1-shift"] \arrow[d] & \prod \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & \lim^{(1)} \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & 0 \\ \prod F(\Sigma x^n) \arrow[r, "1-shift"] & \prod F(\Sigma x^n) \arrow[r] & \lim^{(1)} F(\Sigma x^n) \arrow[r] & 0 \end{tikzcd}.$$

Let show now that the map $\lim \mathcal{T}(x^n, t) \to \lim F(x^n)$ is surjective too. Consider an element $(a_n) \in \lim F(x^n)$, that is, $a_n \in F(x^n)$ such that $a_n = F(\alpha_n)(a_{n+1})$ for all $n \in \mathbb{N}$. We want to construct a commutative diagram

$$\begin{tikzcd} x^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"] & x^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"] & x^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\ t^0 \arrow[r, "\tau_0"'] & t^1 \arrow[r, "\tau_1"'] & t^2 \arrow[r] & \cdots \end{tikzcd}$$

such that the bottom row is a tower in I and $F(f^n)(b^n) = a_n$ for all $n \in \mathbb{N}$. We proceed inductively as follows: $f^0 = 0$ and f^1 comes from the fact that (t^1, b_1) is weak terminal in $\mathcal{P}/F$. Suppose that the construction is done for the first n steps. Further we construct a commutative diagram in $\mathcal{T}$, where the rows are triangles and the second square is homotopy push-out (see [16, Definition 1.4.1 and

Lemma 1.4.4])

$$\begin{tikzcd} p_n \arrow[r] \arrow[d, equal] & x^n \arrow[r, "\alpha_n"] \arrow[d, "f^n"'] & x^{n+1} \arrow[r] \arrow[d] & \Sigma p_n \arrow[d, equal] \\ p_n \arrow[r] & t^n \arrow[r] & y^{n+1} \arrow[r] & \Sigma p_n . \end{tikzcd}$$

By construction $p_n \in \mathcal{P}$, hence $y^{n+1} \in \mathcal{P}^{*(n+1)}$. On the other hand, y^{n+1} is obtained via the triangle

$$x^n \overset{\left(\begin{smallmatrix}\alpha_n \\ -f^n\end{smallmatrix}\right)}{\longrightarrow} x^{n+1} \amalg t^n \to y^{n+1} \to \Sigma x^n;$$

therefore, the sequence

$$F(y^{n+1}) \to F(x^{n+1}) \times F(t^n) \overset{(F(\alpha_n), -F(f^n))}{\longrightarrow} F(x^n)$$

is exact in $\mathcal{A}b$. Because $F(\alpha_n)(a_{n+1}) - F(f^n)(b_n) = a_n - a_n = 0$, we obtain an element $b'_{n+1} \in F(y^{n+1})$ which is sent to (a_{n+1}, b_n) by the first map in the exact sequence above. Thus the two maps constructed in the homotopy push-out square above are actually maps $(x^{n+1}, a_{n+1}) \to (y^{n+1}, b'_{n+1})$, respectively, $(t^n, b_n) \to (y^{n+1}, b'_{n+1})$ in $\mathcal{P}^{*(n+1)}/F$. Since (t^{n+1}, b_{n+1}) is weak terminal in $\mathcal{P}^{*(n+1)}/F$, they can be composed with a map $(y^{n+1}, b'_{n+1}) \to (t^{n+1}, b_{n+1}) \in (\mathcal{P}^{*(n+1)}/F)^{\to}$, in order to obtain a commutative square

$$\begin{tikzcd} x^n \arrow[r, "\alpha_n"] \arrow[d, "f^n"'] & x^{n+1} \arrow[d, "f^{n+1}"] \\ t^n \arrow[r, "\tau_n"'] & t^{n+1} \end{tikzcd}$$

as desired. Denote by $i \in I$ the tower constructed above. We have a composed map $F(t) \to F(t_i) \to \lim F(t^n) \to \lim F(x^n)$ which sends $b \in F(t)$ in turn into b_i, then into $(b_n)_{n\in\mathbb{N}}$ and finally into $(a_n)_{n\in\mathbb{N}}$. This shows that the element $(a_n)_{n\in\mathbb{N}} \in \lim F(x^n) \subseteq \prod F(x^n)$ lifts to an element lying in $\lim \mathcal{T}(x^n, t)$ along the natural map $\prod \mathcal{T}(x_n, t) \to \prod F(x^n)$ which corresponds to b via the Yoneda isomorphism, and the proof of our claim is complete. □

Recall that we say that $\mathcal{T}$ satisfies the Brown representability theorem if every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products is representable.

Theorem 3.7. *Let $\mathcal{T}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a projective class $(\mathcal{P}, \mathfrak{F})$. Suppose also that every category $\mathcal{P}^{*n}/F$ has a weak terminal object, for every $n \in \mathbb{N}$ and every cohomological functor which sends coproducts into products $F : \mathcal{T} \to \mathcal{A}b$. Then $\mathcal{T}$ satisfies the Brown representability theorem.*

Proof. It is shown in [14, Theorem 1.3] that $\mathcal{T}$ satisfies the Brown representability theorem if and only if every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products has a solution object, or equivalently, the category $\mathcal{T}/F$ has a weak terminal object. Thus the conclusion follows from this result corroborated with Proposition 3.6. □

We will say that $\mathcal{T}$ is $\aleph_1$-perfectly generated by a set if it is $\aleph_1$-perfectly generated by a the projective class induced by that set, in the sense above. Thus the theorem above together with Lemma 3.4 give the following corollary.

Corollary 3.8. *Let $\mathcal{T}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a set. Then $\mathcal{T}$ satisfies the Brown representability theorem.*

Remark 3.9. Our condition $\mathcal{T}$ to be $\aleph_1$-perfectly generated by a set is obviously equivalent to the hypothesis of [8, Theorem A]. Therefore, Corollary 3.8 is the same as [8, Theorem A], but with a completely different proof. Note also that every well-generated triangulated category in the sense of Neeman [16] is perfectly generated by a set, in the above sense, as it is shown in [11].

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